Inequality

1. (a) $(m^3 + 1) - (m^2 + m) = (m - 1)^2 (m + 1) > 0$, unless $m = 1$, where equality holds.
$\therefore \quad m^3 + 1 > m^2 + m$.

(b) $(x^5 + y^5) - (x^4y + xy^4) = (x^2 + y^2)(x + y)(x - y)^2 > 0$, unless $x = y$, where equality holds.
$\therefore \quad x^5 + y^5 > x^4y + xy^4$.

(c) $(b + c - a)^2 + (c + a - b)^2 + (a + b - c)^2 - (bc + ca + ab)$
$= 3(a^2 + b^2 + c^2) - 3(ab + bc + ca)$
$= \frac{3}{2}\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right] > 0$, unless $a = b = c$, where equality holds.
$\therefore \quad (b + c - a)^2 + (c + a - b)^2 + (a + b - c)^2 > bc + ca + ab$.

(d) $a + b > 2\sqrt{ab}, \quad b + c > 2\sqrt{bc}, \quad c + a > 2\sqrt{ca}$, equalities hold iff $a = b = c$.
Multiply these three inequalities, we get:
$(a + b)(b + c)(c + a) > 8abc$.

(e) $a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2 - 6abc$
$= a(b - c)^2 + b(c - a)^2 + c(a - b)^2 > 0$, unless $a = b = c$, where equality holds.
$\therefore \quad a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2 > 6abc$.

(f) $\frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b} - 6 = \frac{ab(a+b) + bc(b+c) + ac(b+a) - 6abc}{abc}$
$= \frac{a(b-c)^2 + b(c-a)^2 + c(a-b)^2}{abc} > 0$, unless $a = b = c$, where equality holds.
$\therefore \quad \frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b} > 6$.

(g) $\left(\frac{x}{a} + \frac{y}{b}\right)\left(\frac{a}{x} + \frac{b}{y}\right) - 4 = \frac{ay+bx}{ab} \times \frac{ay+bx}{xy} - 4 = \frac{(ay+bx)^2 - 4(ay)(bx)}{abxy}$
$= \frac{(ay-bx)^2}{abxy} > 0$, unless $ay = bx$, where equality holds.
$\therefore \quad \left(\frac{x}{a} + \frac{y}{b}\right)\left(\frac{a}{x} + \frac{b}{y}\right) > 4$, equality holds $\Leftrightarrow \frac{x}{a} = \frac{y}{b}$.

(h) $(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - 9 = \frac{(a+b+c)(bc+ca+ab) - 9abc}{abc}$
$= \frac{a(b-c)^2 + b(c-a)^2 + c(a-b)^2}{abc} > 0$, unless $a = b = c$, where equality holds.
$\therefore \quad (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) > 9$.

(i) In (h), replace a by l/a, b by b/m, c by c/n, result follows.
Equality holds $\Leftrightarrow \frac{l}{a} = \frac{m}{b} = \frac{n}{c}$.

(j) $x^7 + y^7 - (x^4y^3 + x^3y^4) = (x^4 - y^4)(x^3 - y^3) = (x^2 + y^2)(x + y)(x - y)^2(x^2 + xy + y^2) > 0$
$\therefore \quad x^7 + y^7 > x^4y^3 + x^3y^4$, Equality holds $\Leftrightarrow x = 1$.

(k) $x^7 + 1 - (x^6 + x) = (x^6 - 1)(x - 1) = (x - 1)^2(x^5 + x^4 + x^3 + x^2 + x + 1) > 0$
$\therefore \quad x^7 + 1 > x^6 + x$, Equality holds $\Leftrightarrow x = 1$.

(l) $x^5 + x^{-5} - (x^2 + x^{-2}) = x^5 - x^2 - (x^{-2} - x^{-5}) = x^2(x^3 - 1) - xP^{-5}(x^3 - 1) = (x^3 - 1)(x^2 - x^{-5})$
$= x^{-5}(x^3 - 1)(x^7 - 1) = x^{-5}(x - 1)^2(x^2 + x + 1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1) > 0$
$\therefore \quad x^5 + x^{-5} > x^2 + x^{-2}$, Equality holds $\Leftrightarrow$ $x = 1$.

(m) $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$= \frac{1}{2}(a+b+c)\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right] > 0$, unless $a = b = c$, where equality holds.

$\therefore \quad a^3 + b^3 + c^3 > 3abc$.
Also, $(a^3 + b^3 + c^3)^3 > 27a^3b^3c^3$.
Replace a^3 by a, b^3 by b, c^3 by c,
$\therefore \quad (a + b + c)^3 > 27abc$.

(n) Put $2A = x + y - z$, $2B = y + z - x$, $2C = z + x - y$
The inequality is changed to:
$(A + B)(B + C)(C + A) > 8ABC$ (*)
(1) If A, B, C are all positive, (*) is proved in part (d).
(2) If one of A, B, C is negative,
The R.H.S. of (*) is negative while the L.H.S. is positive. $\therefore$ (*) is always true.
(3) If two or more of the A, B, C are negative, without lost of generality,
let $A, B < 0$.
$\therefore \quad x + y - z < 0$ and $y + z - x < 0$.
Adding the inequalities, we have $y < 0$. Contradict to the given that $y > 0$.

2. Since n is a positive integer, $n + 1 < n + 2 < \ldots < n + n = 2n$ (Equality holds when $n = 1$)

$$\therefore \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} > \frac{1}{2n} + \frac{1}{2n} + \ldots + \frac{1}{2n} = n \times \frac{1}{2n} = \frac{1}{2}$$

3. (1)
$$\frac{1}{n+i} - \frac{1}{n+i+1} = \frac{1}{(n+i)(n+i+1)} < \frac{1}{(n+i)^2}$$

$$\sum_{i=1}^{p} \frac{1}{(n+i)^2} > \sum_{i=1}^{p} \left[\frac{1}{n+i} - \frac{1}{n+i+1}\right] = \sum_{i=1}^{p} \frac{1}{n+i} - \sum_{i=1}^{p} \frac{1}{n+i+1}$$

$$= \left[\frac{1}{n+1} + \sum_{i=2}^{p} \frac{1}{n+i}\right] - \left[\frac{1}{n+p+1} + \sum_{i=1}^{p-1} \frac{1}{n+i+1}\right]$$

$$= \left[\frac{1}{n+1} + \sum_{i=1}^{p-1} \frac{1}{n+i+1}\right] - \left[\frac{1}{n+p+1} + \sum_{i=1}^{p-1} \frac{1}{n+i+1}\right] = \frac{1}{n+1} - \frac{1}{n+p+1}$$

(2)
$$\frac{1}{n+i-1} - \frac{1}{n+i} = \frac{1}{(n+i-1)(n+i)} > \frac{1}{(n+i)^2}$$

$$\sum_{i=1}^{p} \frac{1}{(n+i)^2} < \sum_{i=1}^{p} \left[\frac{1}{n+i-1} - \frac{1}{n+i}\right] = \frac{1}{n} - \frac{1}{n+p}$$

Combining (1) and (2), result follows.

4. $1 = (i + 1) - i = \left(\sqrt{i+1} - \sqrt{i}\right)\left(\sqrt{i+1} + \sqrt{i}\right)$

$$\sqrt{i+1} - \sqrt{i} = \frac{1}{\sqrt{i+1} + \sqrt{i}} < \frac{1}{\sqrt{i} + \sqrt{i}} = \frac{1}{2\sqrt{i}}$$

$$\sum_{i=1}^{n} \left[\sqrt{i+1} - \sqrt{i}\right] < \frac{1}{2} \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$$

$$\sqrt{n+1} - \sqrt{1} < \frac{1}{2} \sum_{i=1}^{n} \frac{1}{\sqrt{i}},$$ result follows.

5. By Mathematical Induction.

Let P(s) be the proposition : $\frac{1}{2\sqrt{s}} < \frac{1}{4^s}C_{2s}^{s} < \frac{1}{\sqrt{2s+1}}$

For P(2), $\frac{1}{2\sqrt{s}} = \frac{1}{2\sqrt{2}} \approx 0.354$, $\frac{1}{4^s}C_{2s}^{s} = \frac{1}{4^2}C_4^2 \approx 0.375$, $\frac{1}{\sqrt{2s+1}} = \frac{1}{\sqrt{5}} \approx 0.447$

$\therefore$ P(2) is true.

Assume P(k) is true for some $k \in \mathbf{N}$, that is, $\frac{1}{2\sqrt{k}} < \frac{1}{4^k}C_{2k}^{k} < \frac{1}{\sqrt{2k+1}}$ …. (1)

For P(k + 1), From (1), multiply throughout by $\frac{2k+1}{2(k+1)} = \frac{(2k+2)(2k+1)}{4(k+1)^2}$,

$$\frac{1}{2\sqrt{k}}\frac{2k+1}{2(k+1)} < \frac{1}{4^k}\frac{(2k+2)(2k+1)}{4(k+1)^2}C_{2k}^{k} < \frac{1}{\sqrt{2k+1}}\frac{2k+1}{2(k+1)}$$

$$\frac{1}{4\sqrt{k}}\frac{2k+1}{(k+1)} < \frac{1}{4^{k+1}}C_{2k+2}^{k+1} < \frac{\sqrt{2k+1}}{2(k+1)} \qquad …. (2)$$

But $\left[\frac{1}{4\sqrt{k}}\frac{2k+1}{(k+1)}\right]^2 = \frac{(2k+1)^2}{[4k(k+1)][4(k+1)]} = \frac{4k^2+4k+1}{4k^2+4k}\frac{1}{4(k+1)} > \frac{1}{4(k+1)}$

And $\left[\frac{\sqrt{2k+1}}{2(k+1)}\right]^2 = \frac{2k+1}{4k^2+8k+4} = \frac{1}{2k+3}\frac{(2k+3)(2k+1)}{4k^2+8k+4} = \frac{1}{2k+3}\frac{4k^2+8k+3}{4k^2+8k+4} < \frac{1}{2k+3}$

From (2), we get: $\frac{1}{2\sqrt{k+1}} < \frac{1}{4^{k+1}}C_{2k+2}^{k+1} < \frac{1}{\sqrt{2k+3}}$

$\therefore$ P(k + 1) is also true.

By the Principle of Mathematical Induction, P(s) is true $s \in \mathbf{N}$.

6. **Method 1**

$$\left(\cot\frac{\theta}{2}-1\right)^2 \ge 0 \Rightarrow \cot^2\frac{\theta}{2} - 2\cot\frac{\theta}{2} + 1 \ge 0 \Rightarrow 2\cot^2\frac{\theta}{2} \ge 2\cot\frac{\theta}{2} + \left(\cot^2\frac{\theta}{2} - 1\right) \qquad ...(1)$$

As $0 < \theta < \pi$, divide (1) by $2\cot\frac{\theta}{2} \ge 0$,

$$\therefore \quad \cot\frac{\theta}{2} \ge 1 + \frac{\cot^2\frac{\theta}{2} - 1}{2\cot\frac{\theta}{2}} = 1 + \cot\theta$$

Method 2

Let $f(\theta) = \cot^2\frac{\theta}{2} - 2\cot\frac{\theta}{2} + 1$, $f'(\theta) = -\frac{1}{2}\csc^2\left(\frac{\theta}{2}\right) + \csc^2\theta$

$$f'\left(\frac{\pi}{2}\right) = -\frac{1}{2}\csc^2\frac{\pi}{4} + \csc^2\frac{\pi}{2} = -\frac{1}{2}\left(\sqrt{2}\right)^2 + 1 = 0$$

$$f''(\theta) = -\frac{1}{2}\left(2\csc\frac{\theta}{2}\right)\left(-\csc\frac{\theta}{2}\cot\frac{\theta}{2}\right)\left(\frac{1}{2}\right) + 2\csc\theta(-\csc\theta\cot\theta) = \frac{1}{2}\csc^2\frac{\theta}{2}\cot\frac{\theta}{2} - 2\cot\theta\cot\theta$$

$$f''\left(\frac{\pi}{2}\right) = \frac{1}{2}\csc^2\frac{\pi}{4}\cot\frac{\pi}{4} - 2\cot\frac{\pi}{2}\cot\frac{\pi}{2} = \frac{1}{2}\times 2\times 1 - 2\times 1\times 0 = 1 > 0$$

$\therefore$ $f(\theta)$ is the minimum when $\theta = \frac{\pi}{2}$.

$\therefore$ $f(\theta) > f\left(\frac{\pi}{2}\right) = 0$, $\cot\frac{\theta}{2} \ge 1 + \cot\theta$, $\forall\theta \in (0, \pi)$

7. $(bc-ad)^2 \geq 0 \quad \Rightarrow \quad b^2c^2 + a^2d^2 \geq 2abcd \quad \Rightarrow \quad b^2c^2 + 2abcd + a^2d^2 \geq 4abcd$

$\therefore \quad (bc + ad)^2 \geq [2\sqrt{(abcd)}]^2$.

As a, b, c, d > 0, taking the positive square root, we have: $bc + ad \geq 2\sqrt{(abcd)}$.

$ab + bc + cd + da \geq ab + 2\sqrt{(abcd)} + cd$

$(a+c)(b+d) \geq \left(\sqrt{ab}+\sqrt{cd}\right)^2$

Take the positive square root again, $\sqrt{(a+c)(b+d)} \geq \sqrt{ab}+\sqrt{cd}$

8. $\frac{a^3+b^3}{2} - \left(\frac{a+b}{2}\right)^3 = \frac{3}{8}(a+b)(a-b)^2 \geq 0$.

9. **(a)** Let d be the common difference of the arithmetic progression.

For $0 \leq k \leq n$,

$$\begin{aligned} a_{k+1}a_{n-k} &= (a_1 + kd)[a + (n-k-1)d] \\ &= a_1^2 + (n-1)a_1d + k(n-k-1)d^2 \\ &\geq a_1^2 + (n-1)a_1d \\ &= a_1[a_1 + (n-1)d] \\ &= a_1a_n \end{aligned}$$

$\therefore \quad \prod_{k=0}^{n-1} a_{k+1}a_{n-k} \geq \prod_{k=0}^{n-1} a_1a_n \quad \Rightarrow \prod_{k=0}^{n-1} a_{k+1} \prod_{k=0}^{n-1} a_{n-k} \geq (a_1a_n)^n \quad \Rightarrow \left(\prod_{k=1}^{n} a_k\right)^2 \geq (a_1a_n)^n$

$\therefore \quad \sqrt{a_1a_n} \leq \sqrt[n]{a_1a_2....a_n}$

(b) For $0 \leq k \leq n$, $\quad a_1 + a_n = a_{k+1} + a_{n-k} \geq 2\sqrt{a_{k+1}a_{n-k}}$

$\therefore \quad \prod_{k=0}^{n-1}(a_1+a_n) \geq \prod_{k=0}^{n-1} 2\sqrt{a_{k+1}a_{n-k}} \quad \Rightarrow (a_1+a_n)^n \geq 2^n\sqrt{\left(\prod_{k=1}^{n} a_k\right)^2}$

$\therefore \quad \sqrt[n]{a_1a_2....a_n} \leq \frac{a_1+a_n}{2}$

Combining (a) and (b), $\sqrt{a_1a_n} \leq \sqrt[n]{a_1a_2....a_n} \leq \frac{a_1+a_n}{2}$

Putting $a_k = k$, for $0 \leq k \leq n$, $\sqrt{1\times n} \leq \sqrt[n]{1.2....n} \leq \frac{1+n}{2} \quad \therefore \quad \sqrt{n} < \sqrt[n]{n!} < \frac{n+1}{2}$.

10. Cauchy-Bunyakovskii-Schwarz (CBS) inequality:

$(a_ix+b)^2 \geq 0 \quad \Rightarrow \sum_{i=1}^{n}(a_ix+b)^2 = \left[\sum_{i=1}^{n}(a_i)^2\right]x^2 + 2\left[\sum_{i=1}^{n}(a_ib_i)\right]x + \left[\sum_{i=1}^{n}(b_i)^2\right] \geq 0$

$\therefore \quad \Delta \leq 0 \Rightarrow \left[\sum_{i=1}^{n}(a_i)^2\right]\left[\sum_{i=1}^{n}(b_i)^2\right] \geq \left[\sum_{i=1}^{n}(a_ib_i)\right]^2$

11. By CBS inequality : $(a_1b_1 + a_2b_2 + ... + a_nb_n)^2 \leq (a_1^2 + a_2^2 + ... + a_n^2)(b_1^2 + b_2^2 + ... + b_n^2)$

Put $b_i = 1 \quad \forall$ i = 1, 2,… , n.

$(a_1 + a_2 + ... + a_n)^2 \leq (a_1^2 + a_2^2 + ... + a_n^2)(n)$

Take the square root, $a_1 + a_2 + ... + a_n \leq \sqrt{n(a_1^2 + a_2^2 + ... + a_n^2)}$

12. **Method 1**

By CBS inequality : $(a_1b_1 + a_2b_2 + ... + a_nb_n)^2 \leq (a_1^2 + a_2^2 + ... + a_n^2)(b_1^2 + b_2^2 + ... + b_n^2)$

Put $a_i^2 = x_i$, $\quad b_i^2 = 1/x_i \quad \forall$ i = 1, 2,… , n.

$$\left(\sqrt{x_1}\frac{1}{\sqrt{x_1}} + \sqrt{x_2}\frac{1}{\sqrt{x_2}} + ... + \sqrt{x_n}\frac{1}{\sqrt{x_n}}\right)^2 \leq (x_1 + x_2 + ... + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n}\right)$$

$\therefore \quad (x_1 + x_2 + ... + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n}\right) \geq n^2$

Method 2

$x_1 + x_2 + ... + x_n \ge n\sqrt[n]{x_1 x_2 .... x_n}$ (A.M. ≥ G.M.)

$\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n} \ge n\sqrt[n]{\frac{1}{x_1}\frac{1}{x_2}...\frac{1}{x_n}}$ (A.M. ≥ G.M.)

Multiply the two inequalities, $(x_1 + x_2 + ... + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n}\right) \ge n^2$

Method 3

(Tchebychef's inequality) If $a_1 \ge a_2 \ge \dots \ge a_n$ and $b_1 \ge b_2 \ge \dots \ge b_n$, then $\frac{\sum a_i}{n}\frac{\sum b_i}{n} \ge \frac{\sum a_i b_i}{n}$.

Without lost of generality, let $x_1 \ge x_2 \ge \dots \ge x_n$ and $y_1 \ge y_2 \ge \dots \ge y_n$,
and put $a_i = x_i$, $b_i = 1/x_i$ for all $i = 1,2, \dots, n$ in Tchebychef's inequality,

$$\frac{\sum x_i}{n}\frac{\sum \frac{1}{x_i}}{n} \ge \frac{\sum x_i \frac{1}{x_i}}{n} = \frac{n}{n} = 1 \qquad \Rightarrow \qquad \sum x_i \sum \frac{1}{x_i} \ge n^2$$

13. **Method 1**

$$\sqrt[n]{n} > \sqrt[n+1]{n+1} \quad \Leftrightarrow \quad (n+1)^n < n^{n+1} \quad \Leftrightarrow \quad \left(\frac{n+1}{n}\right)^n < n \quad \Leftrightarrow \quad \left(1+\frac{1}{n}\right)^n < n$$

Let P(n) be the proposition : $\left(1+\frac{1}{n}\right)^n < n$.

For P(3), $\left(1+\frac{1}{3}\right)^3 = \frac{64}{27} < 3$ $\therefore$ P(3) is true.

Assume P(k) is true for some $k \in \mathbf{N}$. i.e. $\left(1+\frac{1}{k}\right)^k < k$ … (1)

For P(k + 1), $\left(1+\frac{1}{k+1}\right)^{k+1} = \left(1+\frac{1}{k+1}\right)^k\left(1+\frac{1}{k+1}\right) < \left(1+\frac{1}{k}\right)^k\left(1+\frac{1}{k+1}\right) < k\left(1+\frac{1}{k+1}\right)$, by (1)

$$= k + \frac{k}{k+1} < k+1 .$$

$\therefore$ P(k + 1) is also true.
By the Principle of Mathematical Induction, P(n) is true $\forall n \in \mathbf{N}$.

Method 2

Let $f(x) = x^{\frac{1}{x}}$, $\frac{d}{dx}x^{\frac{1}{x}} = \frac{1}{x}x^{\frac{1}{x}-1}\frac{1}{dx}x + (\ln x)x^{\frac{1}{x}}\frac{d}{dx}\frac{1}{x} = x^{\frac{1}{x}-2}(1-\ln x)$

If $x > e$, then $\frac{d}{dx}x^{\frac{1}{x}} < 0$.

$\therefore$ $x^{\frac{1}{x}}$ is decreasing when x is increasing and $x \ge 3$.
Putting x = 3, 4, …., n , n + 1 in f(x), since 3 < 4 < … < n < n + 1, we have:

$$\sqrt[3]{3} > \sqrt[4]{4} > \sqrt[5]{5} > .... > \sqrt[n]{n} > \sqrt[n+1]{n+1} > ....$$

14. **Method 1**

$$\sqrt[n-1]{n} > \sqrt[n]{n+1} \quad \Leftrightarrow \quad n^n > (n+1)^{n-1} \quad \Leftrightarrow \quad n > \left(1+\frac{1}{n}\right)^{n-1}$$

Use Mathematical Induction to prove the last assertion similar to 13 (Method 1).

Method 2

Consider $g(x) = (x+1)^{\frac{1}{x}}$, use the same method as in 13 (Method 2) to show that g(x) is decreasing.

15. For the sequence of positive numbers $1, a, a^2, \ldots, a^{n-1}$, apply A.M. ≥ G.M.

$$\frac{1+a+a^2+\ldots+a^{n-1}}{n} \geq \sqrt[n]{1.a.a^2\ldots.a^{n-1}} \quad\Rightarrow\quad 1+a+a^2+\ldots+a^{n-1} \geq n\left\{a^{\frac{n(n-1)}{2}}\right\}^{\frac{1}{n}}$$

Multiply by $a-1>0$,

$$(a-1)\left(1+a+a^2+\ldots+a^{n-1}\right) \geq na^{\frac{n-1}{2}}(a-1) \quad\Rightarrow\quad a^n - 1 \geq n\left(a^{\frac{n+1}{2}} - a^{\frac{n-1}{2}}\right)$$

16.

Let $1 \leq r \leq n$, $r \in \mathbf{N}$. $(n-r)(r-1) \geq 0 \Rightarrow r(n-r+1) \geq n$. Put $r = 1$, $1.n = n$ Put $r = 2$, $2(n-1) > n$ Put $r = 3$ $3(n-2) > n$: : : : : Put $r = n$ $n.1 = n$ Multiply the inequalities $(n!)^2 > n^n$.	$\frac{2+4+6+\ldots+2n}{n} > \sqrt[n]{2.4.6\ldots(2n)}$, A.M. > G.M. $\frac{1}{n}\frac{(2+2n)n}{2} > \sqrt[n]{2.4.6\ldots(2n)}$ ∴ $2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n) < (n+1)^n$

17. Similar to No. 1(m).

18. $[n-(2r-1)]^2 \geq 0 \quad\Rightarrow\quad n^2 - 2n(2r-1) + (2r-1)^2 \geq 0$

$\Rightarrow\quad n^2 \geq 2n(2r-1) + (2r-1)^2$

$\Rightarrow\quad n^2 \geq [2(n-r+1)-1]\,[2r-1]$

Putting $r = 1, 2, \ldots, n$.

$n^2 \geq [2n-1]\,[1]$

$n^2 \geq [2(n-1)-1]\,[3]$

$n^2 \geq [2(n-3)-1]\,[5]$

: : :

$n^2 \geq [1]\,[2n-1]$

Multiply all inequalities, ∴ $[n^2]^n \geq [1.3.5\ldots(2n-1)]^2$.

19. **<u>Method 1</u>**

Similar to the first part of Number 15 where $a = 2$.

<u>Method 2</u>

Let P(n) be the proposition : $2^n > 1 + n\sqrt{2^{n-1}}$.

For P(2), $2^2 = 4 = 1 + 2 \times 1.5 > 1 + 2\sqrt{2}$ ∴ P(2) is true.

For P(3), $2^3 = 8 = 1 + 3 \times \frac{7}{3} > 1 + 3\sqrt{2^2}$ ∴ P(3) is true.

Assume P(k) is true for some $k \in \mathbf{N}$. i.e. $2^k > 1 + k\sqrt{2^{k-1}}$ … (1)

For P(k + 1), $2^{k+1} = 2 \times 2^k > 2\left(1 + k\sqrt{2^{k-1}}\right)$, by (1)

$= 2 + 2k\sqrt{2^{k-1}} > 1 + 2k\sqrt{2^{k-1}} = 1 + \sqrt{2}k\sqrt{2^k} > 1 + (1 + 0.4)k\sqrt{2^k}$

$= 1 + (k + 0.4k)\sqrt{2^k} > 1 + (k+1)\sqrt{2^k}$, where $k \geq 3$.

∴ P(k + 1) is also true.

By the Principle of Mathematical Induction, P(n) is true $\forall n \in \mathbf{N}$.

20. $\frac{1+2+\ldots+n}{n} \geq \sqrt[n]{1.2\ldots.n} \Rightarrow \frac{n(n+1)}{2n} \geq \sqrt[n]{n!} \Rightarrow \left(\frac{n+1}{2}\right)^n \geq n! \Rightarrow \left(\frac{n+1}{2}\right)^{2n} \geq (n!)^2$ (1)

But $n^n \geq n!$ (2)

Multiply (1) and (2), $n^n\left(\frac{n+1}{2}\right)^{2n} > (n!)^3$.

21. **(i)** Since sum of two sides of a $\Delta \ge$ third side, therefore $(x + y - z), (y + z - x), (z + x - y) \ge 0$.
Apply A.M. $\ge$ G.M.

$$\frac{(x+y-z)+(y+z-x)+(z+x-y)}{3} \ge \sqrt[3]{(x+y-z)(y+z-x)(z+x-y)}$$

$\therefore$ $(x + y + z)^3 > 27(x + y - z)(y + z - x)(z + x - y)$

(ii)
$$\frac{(y+z-x)+(z+x-y)}{2} \ge \sqrt{(y+z-x)(z+x-y)}$$

$\therefore$ $z \ge \sqrt{(y+z-x)(z+x-y)}$ …. (1)

Similarly, $y \ge \sqrt{(x+y-z)(y+z-x)}$ …. (2)

$x \ge \sqrt{(x+y-z)(z+x-y)}$ …. (3)

(1) × (2) × (3), $xyz > (x + y - z)(y + z - x)(z + x - y)$

22. E $= 1!\,3!\,5!\,\ldots(2n-1)! = \dfrac{1!\,2!\,3!\,4!\,5!\ldots(2n-1)!}{2!\,4!\ldots[2(n-1)]!} = \dfrac{1!\,2!\ldots(n-1)!\,n!\,(n+1)!\ldots(2n-1)!}{(2\times1)!\,(2\times2)!\ldots[2(n-1)]!}$

$= \dfrac{1!\,(n+1)!}{(2\times1)!} \times \dfrac{2!\,(n+2)!}{(2\times2)!} \times \ldots \times \dfrac{(n-1)!\,(2n-1)!}{[2(n-1)]!} \times n! = n!\,\prod_{k=1}^{n-1} \dfrac{k!\,(n+k)!}{(2k)!}$

But $\dfrac{k!\,(n+k)!}{(2k)!} = \left[\dfrac{n+1}{k+1}\right]\left[\dfrac{n+2}{k+2}\right]\ldots\left[\dfrac{n+k}{k+k}\right] \times n! \ge n!$, since $n \ge k$.

$\therefore$ E $\ge$ $n!\,\prod_{k=1}^{n-1} n! = n!(n!)^{n-1} = (n!)^n$

23. **(i)** E $= a^2 (p - q)(p - r) + b^2 (q - r)(q - p) + c^2 (r - p)(r - q)$
$= (b^2 + c^2 - 2bc \cos A) (p - q)(p - r) + b^2 (q - r)(q - p) + c^2 (r - p)(r - q)$
$= b^2 [(p - q)(p - r) + (q - r)(q - p)] + c^2 [(p - q)(p - r) + (r - p)(r - q)] - 2bc (p - q)(p - r) \cos A$
$= b^2 (p - q)^2 + c^2 (p - r)^2 - 2bc (p - q)(p - r) \cos A$
$\ge b^2 (p - q)^2 + c^2 (p - r)^2 - 2bc (p - q)(p - r)$, it is trivial that $E \ge 0$ if $(p - q)(p - r) \cos A \le 0$.
$= [b(p - q) - c(p - r)]^2 \ge 0$

(ii) E $= a^2yz + b^2zx + c^2xy$ $= a^2yz + b^2z (-y -z) + c^2 (-y -z)y$ $= (a^2 - b^2 - c^2)yz - b^2z^2 - c^2y^2$
$= -2bc \cos A\, yz - b^2z^2 - c^2y^2$ $\le -2bc\, yz - b^2z^2 - c^2y^2$, it is trivial that $E \le 0$ if $2bc \cos A \ge 0$
$= -(bz + cy)^2 \le 0$

24. $E (a, b, c) = a(a - b)(a - c) + b(b - c)(b - a) + c(c - a)(c - b)$
Since $E(a, b, c) = E(b, a, c) = E(b, c, a) = E(c, a, b) = \ldots$, E is cyclic and symmetric.
Therefore, without lost of generality, let $\mathbf{a \le b \le c}$.
$E (a, b, c)$ $= a(a - b) [(a - b) + (b - c)] + b(b - c)(b - a) + c(c - a)(c - b)$
$= a(a - b)^2 + [a(a - b) (b - c) - b(b - c)(a - b)] + c(c - a)(c - b)$
$= a(a - b)^2 + (a - b)^2 (b - c) + c(c - a)(c - b) \ge 0$, since $c - a \ge 0$, $c - b \ge 0$.

Similarly let $F(a, b, c) = a^2(a - b)(a - c) + b^2(b - c)(b - a) + c^2(c - a)(c - b)$
Since $F(a, b, c) = F(b, a, c) = F(b, c, a) = F(c, a, b) = \ldots$, F is cyclic and symmetric.
$F(a, b, c)$ $= a^2(a - b) [(a - b) + (b - c)] + b^2(b - c)(b - a) + c^2(c - a)(c - b)$
$= a^2(a - b)^2 + [a^2(a - b) (b - c) - b^2(b - c)(a - b)] + c^2(c - a)(c - b)$
$= a^2(a - b)^2 + (a^2 - b^2)(a - b)(b - c) + c^2(c - a)(c - b)$
$= a^2(a - b)^2 + (a + b)(a - b)^2(b - c) + c^2(c - a)(c - b) \ge 0$, since $c - a \ge 0$, $c - b \ge 0$.

25. By CBS inequality, $\left(\sum_{i=1}^{4} x_i^2\right)\left(\sum_{i=1}^{4} y_i^2\right) \ge \left(\sum_{i=1}^{4} x_i y_i\right)^2$

Put $x_1 = \sqrt{a},\ x_2 = \sqrt{b},\ x_3 = \sqrt{c},\ x_4 = \sqrt{d}$; $y_1 = \sqrt{a^3},\ y_2 = \sqrt{b^3},\ y_3 = \sqrt{c^3},\ y_4 = \sqrt{d^3}$

$\therefore$ $(a + b + c + d) (a^3 + b^3 + c^3 + d^3) \ge (a^2 + b^2 + c^2 + d^2)^2$.

26. Let f(x) = x – ln (1 + x) , then $f'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} > 0$, as x > 0. Therefore f(x) is increasing.

Since x > 0 , f(x) > f(0) = 0 – ln (1 + 0) = 0 ⇒ x – ln (1 + x) > 0 ⇒ x > ln (1 + x).

Let $g(x) = \ln(1+x) - \frac{x}{1+x}$, then $g'(x) = \frac{1}{1+x} - \frac{(1+x)-x}{(1+x)^2} = \frac{1}{1+x} - \frac{1}{(1+x)^2} = \frac{1+x-x}{(1+x)^2} = \frac{x}{(1+x)^2} > 0$

Therefore g(x) is increasing.

Since x > 0 , $g(x) > g(0) = \ln(1+0) - \frac{0}{1+0} = 0 \Rightarrow \ln(1+x) > \frac{x}{1+x}$.

27. **Method 1**

$$E = \frac{1-x^n}{n} - \frac{1-x^{n+1}}{n+1} = \frac{(n+1)(1-x^n) - n(1-x^{n+1})}{n(n+1)} = \frac{(1-x^n) - nx^n(1-x)}{n(n+1)}$$

$$= \frac{(1-x)\left[(1+x+x^2+\ldots+x^{n-1}) - nx^n\right]}{n(n+1)} = \frac{(1-x)^2\left[1+2x+3x^2+\ldots+(n-1)x^{n-2}+nx^{n-1}\right]}{n(n+1)} > 0.$$

$$\therefore \quad \frac{1-x^{n+1}}{n+1} < \frac{1-x^n}{n} .$$

Method 2

Since $0 \le x \le 1 \Rightarrow 0 \le x^{n-1} \le 1$ and $0 \le x^n \le 1 \Rightarrow x^n = x\, x^{n-1} \le 1\, x^{n-1} = x^{n-1}$.

Since x^{n-1} and x^n are positive, $\forall x,\ 0 \le x \le 1$

$$x^n \le x^{n-1} \Rightarrow \int_x^1 x^n dx \le \int_x^1 x^{n-1} dx \Rightarrow \left.\frac{x^{n+1}}{n+1}\right|_x^1 \le \left.\frac{x^n}{n}\right|_x^1 \Rightarrow \frac{1-x^{n+1}}{n+1} \le \frac{1-x^n}{n}$$

When x = 1 , equality holds.

When x > 1 , $x^n > x^{n-1} > 1 \Rightarrow$

$$\int_1^x x^n dx > \int_1^x x^{n-1} dx \Rightarrow \left.\frac{x^{n+1}}{n+1}\right|_1^x > \left.\frac{x^n}{n}\right|_1^x \Rightarrow \frac{x^{n+1}-1}{n+1} > \frac{x^n-1}{n} \Rightarrow \frac{1-x^{n+1}}{n+1} < \frac{1-x^n}{n} .$$

28. Let a, m , n > 0 , m > n .

$$\frac{1}{m}\log(1+a^m) < \frac{1}{n}\log(1+a^n) \Leftrightarrow n\log(1+a^m) < m\log(1+a^n) \Leftrightarrow \log(1+a^m)^n < \log(1+a^n)^m$$

$\Leftrightarrow (1+a^m)^n < (1+a^n)^m$, since log x is an increasing function.

$$\Leftrightarrow \frac{(1+a^n)^m}{(1+a^m)^n} > 1 \Leftrightarrow \left[\frac{1+a^n}{1+a^m}\right]^n (1+a^n)^{m-n} > 1$$

If a < 1, then $a^m < a^n \Rightarrow 1+a^m < 1+a^n \Rightarrow (1+a^m)^n < (1+a^n)^m \Rightarrow \frac{(1+a^n)^m}{(1+a^m)^n} > 1$.

If a = 1, then $\frac{(1+a^n)^m}{(1+a^m)^n} = \frac{2^m}{2^n} > 1$.

If a > 1, then $\left[\frac{1+a^n}{1+a^m}\right]^n (1+a^n)^{m-n} > \left[\frac{a^n}{a^m}\right]^n (1+a^n)^{m-n} = a^{(n-m)n}(1+a^n)^{m-n} > a^{n^2-mn}(a^n)^{m-n} = a^0 = 1$.

29. **(a)** $$f(y)=\frac{1}{1-y}-\frac{1}{1+y}-\ln\left(\frac{1+y}{1-y}\right)$$

$$f'(y)=\frac{1}{(1-y)^2}+\frac{1}{(1+y)^2}-\frac{1-y}{1+y}\times\frac{(1-y)-(1+y)(-1)}{(1-y)^2}=\frac{1}{(1-y)^2}+\frac{1}{(1+y)^2}-\frac{2}{(1-y)(1+y)}$$

$$=\frac{(1+y)^2+(1-y)^2-2(1-y^2)}{(1-y)^2(1+y)^2}=\frac{4y^2}{(1-y)^2(1+y)^2}>0.\quad \text{for } 0<y<1.$$

$\therefore$ f(y) is an increasing function.

$$f(y)>f(0)=\frac{1}{1-0}-\frac{1}{1+0}-\ln\left(\frac{1+0}{1-0}\right)=0\Rightarrow\qquad \ln\left(\frac{1+y}{1-y}\right)<\frac{1}{1-y}-\frac{1}{1+y}\ .$$

(b) Put $y=\dfrac{c}{x}$ where $x>c>0$, then $0<y<1$. By (a), $\ln\left(\dfrac{1+c/x}{1-c/x}\right)<\dfrac{1}{1-c/x}-\dfrac{1}{1+c/x}$

Therefore $0<\ln\left(\dfrac{x+c}{x-c}\right)<\dfrac{x}{x-c}-\dfrac{x}{x+c}$. For $a>b>c$, we have

$$\int_b^a \ln\left(\frac{x+c}{x-c}\right)dx<\int_b^a\left(\frac{x}{x-c}-\frac{x}{x+c}\right)dx\Rightarrow\int_b^a \ln(x+c)dx-\int_b^a \ln(x-c)dx<\int_b^a\frac{xdx}{x-c}-\int_b^a\frac{xdx}{x+c}$$

$$\Rightarrow\int_b^a\left[\ln(x+c)+\frac{x}{x+c}\right]dx<\int_b^a\left[\ln(x-c)+\frac{x}{x-c}\right]dx\Rightarrow x\ln(x+c)\Big|_b^a<x\ln(x-c)\Big|_b^a$$

$\Rightarrow$ a ln (a + c) – b ln (b + c) < a ln (a – c) – b ln (b – c)

$\Rightarrow$ a ln (a + c) –a ln (a – c) < b ln (b + c) – b ln (b – c)

$$\Rightarrow a\ln\left(\frac{a+c}{a-c}\right)<b\ln\left(\frac{b+c}{b-c}\right)\Rightarrow\ln\left(\frac{a+c}{a-c}\right)^a<\ln\left(\frac{b+c}{b-c}\right)^b\quad\Rightarrow\quad\left(\frac{a+c}{a-c}\right)^a<\left(\frac{b+c}{b-c}\right)^b$$

30. Put $\alpha=\beta+\gamma$

$$\left(1+\frac{1}{\beta}\right)^\beta=1^\gamma\left(1+\frac{1}{\beta}\right)^\beta<\left[\frac{\gamma(1)+\beta(1+1/\beta)}{\gamma+\beta}\right]^{\gamma+\beta}\quad\text{, by A.M. > G.M.}$$

$$=\left[\frac{\gamma+\beta+1}{\gamma+\beta}\right]^{\gamma+\beta}=\left(\frac{\alpha+1}{\alpha}\right)^\alpha=\left(1+\frac{1}{\alpha}\right)^\alpha$$

$\therefore$ $S_n=\left(1+\dfrac{1}{n}\right)^n$ is a monotonic increasing function.

For $n>1$, $S_n=\left(1+\dfrac{1}{n}\right)^n>\left(1+\dfrac{1}{1}\right)^1=2$. Also,

$$S_n=1+\binom{n}{1}\frac{1}{n}+\binom{n}{2}\frac{1}{n^2}+\dots+\binom{n}{r}\frac{1}{n^r}+\dots=1+n\left(\frac{1}{n}\right)+\frac{n(n-1)}{2!}\left(\frac{1}{n^2}\right)+\dots+\frac{n(n-1)\dots(n-r+1)}{r!}\left(\frac{1}{n^r}\right)+\dots$$

$$<1+1+\frac{n\times n}{2!}\left(\frac{1}{n^2}\right)+\dots+\frac{n\times n\times\dots\times n}{r!}\left(\frac{1}{n^r}\right)+\dots=1+1+\frac{1}{2!}+\frac{1}{3!}+\dots+\frac{1}{r!}+\dots$$

< 1 + 1 + 0.5 + 0.16667 + 0.04167 + 0.00833 + 0.00138 +…

= 2.718…

31. $$2^m+4^m+6^m+\dots+(2n)^m\ge n(n+1)^m\Leftrightarrow 1^m+2^m+3^m+\dots+n^m\ge n\left(\frac{n+1}{2}\right)^m$$

By Tchebychev's inequality,

$$n\left(1^{m-1}\times1+2^{m-1}\times2+3^{m-1}\times3+\dots+n^{m-1}\times n\right)\ge\left(1^{m-1}+2^{m-1}+3^{m-1}+\dots+n^{m-1}\right)(1+2+3+\dots+n)$$

$$=\left(1^{m-1}+2^{m-1}+3^{m-1}+\dots+n^{m-1}\right)\left[\frac{n(n+1)}{2}\right]\qquad\dots.(1)$$

$$n\left(1^{m-2}\times 1+2^{m-2}\times 2+3^{m-2}\times 3+\ldots+n^{m-2}\times n\right)\ge\left(1^{m-2}+2^{m-2}+3^{m-2}+\ldots+n^{m-2}\right)\left[\frac{n(n+1)}{2}\right] \qquad \ldots.(2)$$

: : : : : :

$$n\left(1\times 1+2\times 2+3\times 3+\ldots+n\times n\right)\ge\left(1+2+3+\ldots+n\right)\left[\frac{n(n+1)}{2}\right] \qquad \ldots(n)$$

Multiply the inequalities,

$$n^{m-1}\left(1^m+2^m+3^m+\ldots+n^m\right)\ge\left(\frac{n(n+1)}{2}\right)\left(\frac{n(n+1)}{2}\right)^{m-1}=n^m\left(\frac{n+1}{2}\right)^m$$

Therefore $1^m+2^m+3^m+\ldots+n^m\ge n\left(\frac{n+1}{2}\right)^m$

32. By Tchebychev's inequality, $\frac{a^4+b^4+c^4}{3}=\frac{a^2a^2+b^2b^2+c^2c^2}{3}\ge\frac{a^2+b^2+c^2}{3}\frac{a^2+b^2+c^2}{3}$

$$=\left(\frac{aa+bb+cc}{3}\right)^2\ge\left(\frac{a+b+c}{3}\frac{a+b+c}{3}\right)^2=\left(\frac{a+b+c}{3}\right)^4$$

$\therefore$ $27(a^4+b^4+c^4)>(a+b+c)^4$.

33. **Method 1**

$1^3+2^3+3^3+\ldots.+n^3=\frac{1}{4}n^2(n+1)^2$

$8(1^3+2^3+3^3+\ldots.+n^3)-n(n+1)^3=8\times\frac{1}{4}n^2(n+1)^2-n(n+1)^3=n(n+1)^2[2n-n-1]=n(n+1)^2(n-1)$

≥ 0, $n>1$.

$\therefore$ $n(n+1)^3\le 8(1^3+2^3+3^3+\ldots.+n^3)$.

Method 2

$8k^3-[k(k+1)^3-(k-1)k^3]=8k^3-k[k^3+3k^2+3k+1-k^3+k^2]=4k^3-3k^2-k=k(4k+1)(k-1)\ge 0$

$\therefore$ $k(k+1)^3-(k-1)k^3\le 8k^3$ $\qquad \sum_{k=1}^{n}\left[k(k+1)^3-(k-1)k^3\right]\ge 8\sum_{k=1}^{n}k^3$

$\therefore$ $n(n+1)^3\le 8(1^3+2^3+3^3+\ldots.+n^3)$.

34. By A.M. > G.M., $\frac{x_1+x_2+x_3}{3}\ge\sqrt[3]{x_1x_2x_3}$ …. (1)

$$\frac{\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}}{3}\ge\sqrt[3]{\frac{1}{x_1}\frac{1}{x_2}\frac{1}{x_3}} \qquad \ldots. \quad (2)$$

9×(1)×(2), $(x_1+x_2+x_3)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}\right)\ge 9$

35. **(i)** Put $f(x)=e^x-(1+x)$, then $f'(x)=e^x-1$.

If $x>0$, $f'(x)>e^0-1=0$. $\therefore$ f(x) is increasing.

If $x<0$, $f'(x)<e^0-1=0$. $\therefore$ f(x) is decreasing.

$\therefore$ f(x) is a minimum at $x=0$.

$\therefore$ $f(x)=e^x-(1+x)>f(0)=e^0-(1+0)=0$ and $e^x>1+x$ for $x\ne 0$.

(ii) Put $g(x)=\tan x-\left(x+\frac{x^3}{3}\right)$, $x\in\left(0,\frac{\pi}{2}\right)$, then $g'(x)=\sec^2 x-1-x^2=\tan^2 x-x^2>0$,

since $\tan x>x$, for $x\in\left(0,\frac{\pi}{2}\right)$.

$\therefore$ g(x) is increasing. Since x > 0, $g(x) = \tan x - \left(x + \dfrac{x^3}{3}\right) > g(0) = \tan 0 - \left(0 + \dfrac{0^3}{3}\right) = 0.$

$\therefore \quad x + \dfrac{x^3}{3} < \tan x, \quad x \in \left(0, \dfrac{\pi}{2}\right)$.

36. If $b^2 < ac$, then $x^2 + 2bx + ac > 0 \quad \forall x \in \mathbb{R}$ and $a \neq 0$.

Put x = 2a, $a(4a + 4b + c) > 0$ …. (1)

Put x = a, $a(a + 2b + c) > 0$ …. (2)

(1) × (2), $a^2(4a + 4b + c)(a + 2b + c) > 0$.

Since $a^2 > 0$ and $a + 2b + c > 0$, we have $4a + 4b + c > 0$.

37. $$C_k^n \frac{1}{n^k} = \frac{n!}{k!(n-k)!}\frac{1}{n^k} = \frac{1}{k!} \times \frac{n}{k} \times \frac{n-1}{k} \times \frac{n-2}{k} \times \ldots \times \frac{n-k+1}{k} = \frac{1}{k!} \times 1 \times \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \ldots \left(1 - \frac{k+1}{n}\right) \le \frac{1}{k!}$$

$$\left(1 + \frac{1}{n}\right)^n = 1 + C_1^n \frac{1}{n} + C_2^n \frac{1}{n^2} + \ldots + C_k^n \frac{1}{n^k} + \ldots + C_n^n \frac{1}{n^n} \le 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{k!} + \ldots + \frac{1}{n!}$$

$$< 1 + 1 + \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \ldots + \frac{1}{(n-2)(n-1)} + \frac{1}{(n-1)n}$$

$$= 1 + 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots \left(\frac{1}{n-2} - \frac{1}{n-1}\right) + \left(\frac{1}{n-1} - \frac{1}{n}\right) = 1 + 1 + 1 - \frac{1}{n} < 3$$

38. **(i)** $2(x^2 + y^2) - (x + y)^2 = x^2 - 2xy + y^2 = (x - y)^2 \quad \ge \quad 0$

$\therefore \quad 2(x^2 + y^2) \ge (x + y)^2$ and the equality holds $\Leftrightarrow (x - y)^2 = 0 \Leftrightarrow x = y$.

(ii) $$2\left[\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2\right] \underset{\text{by (a)}}{\ge} \left(a + \frac{1}{a} + b + \frac{1}{b}\right)^2 = \left(1 + \frac{1}{a} + \frac{1}{b}\right)^2 = \left(1 + \frac{a+b}{ab}\right)^2$$

$$\ge \left(1 + \left(\frac{2}{a+b}\right)^2\right)^2, \qquad \text{since} \left(\frac{a+b}{2}\right)^2 \ge ab.$$

$$= \left[1 + 2^2\right]^2 = 25 \qquad \therefore \qquad \left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 \ge \frac{25}{2}.$$

39. **(i)** **(1)** If $x \ge 1$, $x^2 - \dfrac{1}{x^3} = \dfrac{x^5 - 1}{x^3} \ge 0$, since $x^5 \ge 1$.

(2) If $0 < x < 1$, $x^2 - \dfrac{1}{x^3} = \dfrac{x^5 - 1}{x^3} \le 0$, since $x^5 \le 1$.

Combining (1) and (2), $(x-1)\left(x^2 - \dfrac{1}{x^3}\right) \ge 0 \Rightarrow x^2(x-1) \ge \dfrac{x-1}{x^3} \Rightarrow x^3 - x^2 \ge \dfrac{1}{x^2} - \dfrac{1}{x^3}$

The equality holds when x = 1.

(ii) Let a, ar, ar^2 be the sides of the triangle.

Since the sum of any two sides > the third side.

$ar + ar^2 > a$ …. (1)

$ar + a > ar^2$ …. (2)

From (1), Since $a > 0$, $r + r^2 > 1$, $r^2 + r - 1 > 0$ $\quad \therefore \quad r > \dfrac{\sqrt{5} - 1}{2}$ or $r < \dfrac{-\sqrt{5} - 1}{2}$ ….(3)

Since $r > 0$, $\quad \therefore \quad r > \dfrac{\sqrt{5} - 1}{2}$

From (2), Since $a > 0$, $r^2 - r - 1 < 0$. $\quad \therefore \quad r < \dfrac{1 + \sqrt{5}}{2}$ or $r > \dfrac{\sqrt{5} - 1}{2}$ …. (4)

Combining (3) and (4), $\frac{\sqrt{5}-1}{2} < r < \frac{\sqrt{5}+1}{2}$

40. $x < 1$ or $x > 2$.

41. $\tan\frac{A}{2} = \tan\frac{\pi-(B+C)}{2} = \tan\left(\frac{\pi}{2} - \frac{B+C}{2}\right) = \cot\frac{B+C}{2} = \frac{1-\tan\frac{B}{2}\tan\frac{C}{2}}{\tan\frac{B}{2}+\tan\frac{C}{2}}$

$\Rightarrow \quad \tan\frac{A}{2}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right) = 1-\tan\frac{B}{2}\tan\frac{C}{2}$

$\Rightarrow \quad \tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2} = 1$ …. (1)

Now, $\left(\tan\frac{A}{2}-\tan\frac{B}{2}\right)^2 + \left(\tan\frac{B}{2}-\tan\frac{C}{2}\right)^2 + \left(\tan\frac{C}{2}-\tan\frac{A}{2}\right)^2 \geq 0$

$\Rightarrow \quad 2\left(\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\right) - 2\left(\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}\right) \geq 0$

$\Rightarrow \quad 2\left(\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\right) - 2(1) \geq 0$

$\Rightarrow \quad \tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2} \geq 1$

42. Since $\left(\frac{a+b}{2}\right)^{a+b} > \left(\sqrt{ab}\right)^{a+b} = (ab)^{\frac{a+b}{2}}$

It remains to show $(ab)^{\frac{a+b}{2}} > a^b b^a$ …. (1)

Since (1) is symmetric in a, b we can let $a \geq b > 0$.

$\frac{(ab)^{\frac{a+b}{2}}}{a^b b^a} = \left(\frac{a}{b}\right)^{\frac{a-b}{2}} > 1$, since $\frac{a}{b} \geq 1$, $\frac{a-b}{2} \geq 0$.

Result follows.